Counterexamples to containment problems for fat points schemes in the projective plane via multiplier ideals

Víctor González-Alonso, Piotr Pokora

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Abstract

In this note we address the relation between symbolic and ordinary powers of the ideal of a reduced set or points in projective space: the so-called containment problem. In particular, we obtain sharp lower bounds on the Waldschmidt constants of ideals of points in the plane not satisfying $I^{(2r-1)} \subseteq I^r$.

Keywords containment problems, line configurations, multiplier ideals, symbolic powers of homogeneous ideals

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1 Preliminaries

In this paper we consider the planar case of containment problems – for a more general introduction to the subject we refer to the very recent survey [11]. Let $\mathcal{P} = \{P_1, ..., P_s\}$ be a finite set of mutually distinct points in the projective plane over an algebraically closed field $\mathbb{K}$ of characteristic 0. Now we can define the (homogeneous) ideal associated to $\mathcal{P}$ by

$$I := I(\mathcal{P}) = I(P_1) \cap ... \cap I(P_s) \subset \mathbb{K}[x, y, z]$$

where $I(P_i) \subset \mathbb{K}[x, y, z]$ is the defining ideal of P_i. Then for $I(\mathcal{P})$ the m-th symbolic power with $m \geqslant 1$ is given by

$$I^{(m)} = I(P_1)^m \cap ... \cap I(P_s)^m,$$

that is, the ideal generated by homogeneous forms vanishing to order $\geqslant m$ at each point P_i. Note that this differs from the original definition of symbolic power, and the equivalence of both definitions is a consequence of the celebrated Nagata-Zariski Theorem. For any $m \geqslant 1$ the inclusion $I^m \subseteq I^{(m)}$ holds trivially, but for $m \geqslant 2$ it is usually very far from being an equality. Determining some kind of reverse inclusions (with different exponents) is one of the most intriguing problems devoted to symbolic and ordinary powers of homogeneous ideals: the so-called containment problem.

Problem 1.1. Decide for which pairs of positive integers (m, r) the following containment holds:

$$I^{(m)} \subseteq I^r.$$

It was somehow surprising that it is possible to show a simple and uniform statement for the containment problem. Let us recall that Ein, Lazarsfeld, and Smith [5] in characteristic 0, and Hochster and Huneke [7] in positive characteristic, showed in particular the following (with analogous definitions of I and $I^{(m)}$ for a finite set $\mathcal{P}$ in $\mathbb{P}^n$).

Theorem 1.2. *Let $\mathcal{P}$ be a finite set of points in $\mathbb{P}^n$ and denote by $I = I(\mathcal{P})$ the associated radical ideal. Then for $r \geqslant 1$ one has*

$$I^{(nr)} \subseteq I^r.$$

Simultaneously, Huneke asked whether the above uniform containment is sharp. In particular, one can formulate the following problem [11, Problem 1.5].

Problem 1.3. (Bocci, Harbourne, Huneke) Let $\mathcal{P} \subset \mathbb{P}^2$ be a finite set of points in the projective plane and denote by $I = I(\mathcal{P})$ the associated radical ideal. Is it true that for $r \geqslant 1$ one has

$$I^{(2r-1)} \subseteq I^r? \tag{1}$$

In particular, it can be asked whether the containment

$$I^{(3)} \subseteq I^2 \tag{2}$$

holds (for $n = 2$). It came as a surprise when Dumnicki, Szemberg, and Tutaj-Gasińska [4] provided the very first counterexample to (2) over the complex numbers. Their counterexample is based on the singular locus of the dual-Hesse configuration of 9 lines and 12 triple points as singular points. It is worth pointing out here that the theory of line configurations plays an important role in this subject – most known counterexamples are based on singular loci of interesting line arrangements. For instance, Klein's and Wiman's configurations of lines [8, 12] provide other counterexamples to (2) – for more details please consult [1, 2, 10].

In general, it is difficult to find counterexamples to the containment (2). One of the reasons might be that there are not numerical criterions which would allow to decide whether a certain configuration of points is a candidate for the failure of (2). Up to now one of the most useful results in this context is due to Bocci and Harbourne – they showed in particular [3, Lemma 2.3.3] that if I is a homogeneous radical ideal in $\mathbb{K}[x, y, z]$, then

$$I_d^{(2r-1)} \subseteq I_d^{(r)} = I_d^r,$$

for any $d \geqslant \text{reg}(I^r)$, where $\text{reg}(I)$ denotes the Castelnuovo-Mumford regularity of I.

This result provides a strategy to attack the containment problem (2), namely in order to find a counterexample the first step is to find a form in $I^{(3)}$ of degree lower than $\text{reg}(I^2)$. It is easy to observe that in the case of the dual-Hesse configuration we start with the form of degree 9 defining the configuration and $\text{reg}(I^2) = 10$, so this actually gave a possibility to construct a counterexample.

In the present paper we propose a numerical condition necessary for a given homogeneous radical ideal I to violate the containment (2). In order to formulate the result, let us recall the following object.

Definition 1.4. Let $I \leqslant \mathbb{K}[x_0, \ldots, x_n]$ be a radical homogeneous ideal. Then the initial degree of I is defined as

$$\alpha(I) = \min\{t : I_t \neq 0\}.$$

Definition 1.5. Let $I \leqslant \mathbb{K}[x_0, \ldots, x_n]$ be a radical homogeneous ideal. The Waldschmidt constant of I is defined as

$$\widehat{\alpha}(I) = \inf_{m \geqslant 1} \frac{\alpha(I^{(m)})}{m} = \lim_{m \to \infty} \frac{\alpha(I^{(m)})}{m}.$$

We can now state the main result of this note:

Main result. *Let $\mathcal{P}$ be a finite set of mutually distinct points in the projective plane over the field $\mathbb{K}$, and denote by $I := I(\mathcal{P})$ the associated radical ideal. If $I^{(2r-1)} \nsubseteq I^r$, then one has*

$$2r\widehat{\alpha}(I) \geqslant \alpha(I^{(2r-1)}) + 3,$$

and in particular $\widehat{\alpha}(I) \geqslant 3$.

This result provides us, when the containment (1) does not hold, with a non-trivial lower bound on the Waldschmidt constant of I.

2 Proof of the Main Result

We will introduce the tools used by Ein Lazarsfeld and Smith to prove Theorem 1.2, that is, (asymptotic) multiplier ideals. For more details, please consult [5]. Since these techniques apply to any dimension and for the sake of generality, we will consider finite sets of points in a projective space of arbitrary dimension n. We also assume from now on that $\mathbb{K}=\mathbb{C}$.

Let X be a smooth complex algebraic variety and $\mathfrak{a}\subset\mathcal{O}_X$ an ideal sheaf. Recall that a log-resolution of $\mathfrak{a}$ is a proper birational morphism $\pi: X'\to X$ whose exceptional set $E=Exc(\pi)$ is a divisor and such that $\mathfrak{a}\mathcal{O}_{X'}=\mathcal{O}_{X'}(-F)$ for an effective divisor F such that $E+F$ has simple normal crossings.

Definition 2.1. With the above notations, the multiplier ideal (sheaf) of $\mathfrak{a}$ of exponent $c\in\mathbb{R}_{\geqslant 0}$ is

$$\mathcal{J}(\mathfrak{a}^c)=\mathcal{J}(c\cdot\mathfrak{a})=\pi_*\mathcal{O}_{X'}(K_{X'/X}-\lfloor cF\rfloor)\subseteq\mathcal{O}_X,$$

where $K_{X'/X}=K_{X'}-\pi^*K_X$ is the relative canonical divisor of the log-resolution.

The above definition is independent of the chosen log-resolution of $\mathfrak{a}$.

Let now $\{\mathfrak{a}_d\}_{d\in\mathbb{Z}_{\geqslant 1}}$ be a multiplicative graded system of ideals in $\mathcal{O}_X$, that is, a sequence of ideals such that $\mathfrak{a}_d\mathfrak{a}_e\subseteq\mathfrak{a}_{d+e}$ for all $d,e\geqslant 1$. Then for all $c\in\mathbb{R}_{\geqslant 0}$ and any integers $d,m\geqslant 1$ it holds

$$\mathcal{J}(\mathfrak{a}_d^c)\subseteq\mathcal{J}(\mathfrak{a}_{md}^{\frac{c}{m}}). \tag{3}$$

Since $\mathcal{O}_X$ is a sheaf of noetherian rings, for a fixed c and d the ideals in (3) admit a *supremum*, or even a limit ideal: the asymptotic multiplier ideal of the subsystem $\{\mathfrak{a}_{md}\}_{m\geqslant 1}$, denoted as

$$\mathcal{J}(||\mathfrak{a}_d||^c)=\mathcal{J}(c\cdot||\mathfrak{a}_d||):=\sup_{m\geqslant 1}\mathcal{J}(\mathfrak{a}_{md}^{\frac{c}{m}}).$$

Asymptotic multiplier ideals satisfy the following inclusions (see [5, Proposition 1.7]):

1. $\mathfrak{a}_d\subset\mathcal{J}(||\mathfrak{a}_d||)$ and
2. $\mathcal{J}(||\mathfrak{a}_{md}||)\subseteq\mathcal{J}(||\mathfrak{a}_d||)^m$.

If furthermore $\mathcal{J}(||\mathfrak{a}_d||)\subseteq\mathfrak{a}_1$, then it follows that $\mathfrak{a}_{md}\subseteq\mathfrak{a}_1^m$ for any $m\geqslant 1$ (see [5, Theorem 2.1]).

Let us now go back to our geometric situation, where $I\subseteq\mathbb{C}[x_0,\dots,x_n]$ is the radical homogeneous ideal defining a finite set of points $\{P_1,\dots,P_s\}\subset\mathbb{P}^n$. For brevity we will denote the initial degree of the m-th symbolic power of I by $\alpha_m=\alpha_m(I):=\alpha(I^{(m)})$, and the Waldschmidt constant of I by $\widehat{\alpha}$.

By taking affine cones, I corresponds to the ideal (sheaf) of a finite set of lines $\{l_1,\dots,l_s\}$ through the origin in $\mathbb{C}^{n+1}$. Let $\pi: X'\longrightarrow\mathbb{C}^{n+1}$ be the log-resolution of I obtained by first blowing up the origin, with exceptional divisor $E\cong\mathbb{P}^n$, and then blowing up the strict transforms of the lines $l_i\subset\mathbb{C}^{n+1}$, with exceptional divisors $F_1,\dots,F_s$. Let F_0 be the strict transform of E by this second blow-up, which is of course isomorphic to the blow-up $Bl_{P_1,\dots,P_s}(\mathbb{P}^n)$. Then almost by definition we have

$$I^{(m)}=\pi_*\mathcal{O}_{X'}(-m(F_1+\dots+F_s)).$$

Let D_m denotes the effective divisor on X' such that $I^{(m)}\mathcal{O}_{X'}=\mathcal{O}_{X'}(-D_m)$. This is the common part to all the divisors $\pi^*(f)$, where $f\in I^{(m)}$. Since every such f has degree at least α_m and vanishes at P_i's to order at least m, we trivially have

$$D_m\geqslant\alpha_mF_0+m(F_1+\dots+F_s). \tag{4}$$

By definition of α_m, there exists some $f\in I^{(m)}$ of degree exactly α_m. Moreover, if $f_1,\dots,f_s$ are linear forms such that $f_i(P_i)=0$ but $f_i(P_j)\neq 0$ for any $i\neq j$, then $(f_1\cdot\ldots\cdot f_s)^m\in I^{(m)}$ vanishes at the P_i's to order exactly m. Summing up, this shows that the inequality (4) is indeed an equality, i.e., one has

$$I^{(m)}\mathcal{O}_{X'}=\mathcal{O}_{X'}(-\alpha_mF_0-m(F_1+\dots+F_s)). \tag{5}$$

Lemma 2.2. *For any $0 \leqslant d \leqslant \alpha_m$ one has*

$$\pi_* \mathcal{O}_{X'}(-dF_0 - m(F_1 + ... + F_s)) = I^{(m)},$$

while $\pi_* \mathcal{O}_{X'}(-dF_0 - m(F_1 + ... + F_s)) \subsetneq I^{(m)}$ for any $d > \alpha_m$.

Proof. Note that any form vanishing at the P_i's to order $\geqslant m$ vanishes at 0 with multiplicity at least α_m. $\square$

As it was observed in [5], the family

$$\{I^{(m)}\}_{m \geqslant 1}$$

is a multiplicative graded system of ideals with $I^{(1)} = I$. Thus, by the above discussion, if

$$\mathcal{J}(||I^{(d)}||) \subseteq I, \tag{6}$$

for some $d \geqslant$, then $I^{(dr)} \subseteq I^r$ for every $r \geqslant 2$.

Now we would like to establish a condition when the inclusion $I^{(dr-1)} \subseteq I^r$ of the previous symbolic power also holds. To this aim we first need the following more explicit expression for the asymptotic multiplier ideals:

Lemma 2.3. *With the above notations, it holds*

$$\mathcal{J}(||I^{(m)}||) = \pi_* \mathcal{O}_{X'} \left(- \left(\lfloor m\widehat{\alpha} \rfloor - n \right) F_0 - (m - n + 1)(F_1 + \cdots + F_s) \right).$$

Proof. It follows from direct computations. Firstly

$$\mathcal{J}(||I^{(m)}||) = \lim_{N \to \infty} \mathcal{J}\left(\frac{1}{N} \cdot I^{(Nm)}\right) = \lim_{N \to \infty} \pi_* \mathcal{O}_{X'}\left(K_{X'} - \left\lfloor \frac{1}{N} D_{Nm} \right\rfloor\right),$$

where $K_{X'} = K_{X'/\mathbb{C}^{n+1}} = nF_0 + (n-1)(F_1 + ... + F_s)$ and D_m is the effective divisor such that

$$I^{(m)} \mathcal{O}_{X'} = \mathcal{O}_{X'}(-D_m),$$

i.e. after (5),

$$D_m = \alpha_m F_0 + m(F_1 + ... + F_s).$$

Thus

$$K_{X'} - \left\lfloor \frac{1}{N} D_{Nm} \right\rfloor = \left(n - \left\lfloor \frac{\alpha_{Nm}}{N} \right\rfloor\right) F_0 + (n - 1 - m)(F_1 + ... + F_s).$$

Finally, note that the fact that $\widehat{\alpha} = \lim_{N \to \infty} \frac{\alpha_{Nm}}{N} = \inf_{N \geqslant 0} \frac{\alpha_{Nm}}{N}$ implies

$$\lim_{N \to \infty} \left\lfloor \frac{\alpha_{Nm}}{N} \right\rfloor = \left\lfloor \lim_{N \to \infty} \frac{\alpha_{Nm}}{N} \right\rfloor = \left\lfloor m \lim_{N \to \infty} \frac{\alpha_{Nm}}{Nm} \right\rfloor = \lfloor m\widehat{\alpha} \rfloor,$$

which completes the proof. $\square$

It is now inmediate to prove the following

Proposition 2.4. *Keeping the above notations we have*

$$\mathcal{J}(||I^{(m)}||) \subseteq I^{(m+1-n)}$$

for any $m \geqslant n$. Furthermore, equality holds if and only if $\lfloor m\widehat{\alpha} \rfloor - n \leqslant \alpha_{m+1-n}$.

Note that in particular it always holds $\mathcal{J}(||I^{(n)}||) \subseteq I$, which allows to prove Theorem 1.2. Indeed, by the aforementioned properties of multiplier ideals, it always holds

$$I^{(nr)} \subseteq \mathcal{J}(||I^{(nr)}||) \subseteq \mathcal{J}(||I^{(n)}||)^r \subseteq I^r.$$

Now we go back to the containment problem for smaller symbolic exponents of points in the projective plane, i.e. we set $n = 2$. Proposition 2.4 shows that $\mathcal{J}(||I^{(2r)}||) \subseteq I^{(2r-1)}$, and equality would imply

$$I^{(2r-1)} \subseteq \mathcal{J}(||I^{(2r)}||) \subseteq \mathcal{J}(||I^{(2)}||)^r \subseteq I^r.$$

Thus a necessary condition to have $I^{(2r-1)} \not\subseteq I^r$ is that $\mathcal{J}(||I^{(2r)}||) \subsetneq I^{(2r-1)}$, or equivalently after Proposition 2.4

$$\lfloor 2r\widehat{\alpha} \rfloor - 2 > \alpha_{2r-1}.$$

This finally implies that

$$2r\widehat{\alpha} \geqslant \alpha_{2r-1} + 3,$$

which is the first claim of the Main Result.

The last claim follows by simply recalling that $\widehat{\alpha} \leqslant \frac{\alpha_{2r-1}}{2r-1}$, and the proof is now complete.

Remark 2.5. As it is observed in [2, Page 6], one always has

$$\frac{\alpha_m}{m+1} \leqslant \widehat{\alpha} \leqslant \frac{\alpha_m}{m}.$$

In particular, for $m = 3$ one has

$$\widehat{\alpha} \geqslant \frac{\alpha_3}{4}.$$

Now, if $I^{(3)} \not\subset I^2$, then using our necessary conditions one obtains the stronger bound

$$\widehat{\alpha} \geqslant \frac{\alpha_3 + 3}{4}.$$

Remark 2.6. Now we would like to give an example showing that our bound is sharp. Consider the dual-Hesse configuration $\mathcal{H}$ of 9 lines and 12 triple points. As we can see [9, Example 4.4], the Waldschmidt constant for the radical ideal I associated to $\mathcal{H}$ is equal to 3. Observe that $\alpha_3 = 9$ and then

$$3 = \widehat{\alpha} \geqslant \frac{\alpha_3 + 3}{4} = \frac{9+3}{4} = 3.$$

Remark 2.7. In dimension $n \geqslant 3$, the same construction allows to prove that if $I^{(nr-n+1)} \not\subseteq I^r$, then

$$nr\widehat{\alpha} \geqslant \alpha_{nr-n+1} + n + 1$$

In particular, it must hold $\widehat{\alpha} \geqslant \frac{n+1}{n-1}$, which is also less restrictive for bigger n. However, it provides no consequence of the non-containment of the intermediate symbolic powers $I^{(nr-n+2)}, \dots, I^{(nr-1)}$ in I^r. It is worth pointing out that the above question about the containment was formulated as a rather challenging problem by Harbourne and Huneke [6, Conjecture 4.1.1].

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Víctor González-Alonso, Institut für Algebraische Geometrie, and Riemann Center for Geometry and Physics, Leibniz Universität Hannover, Welfengarten 1, D-30167 Hannover, Germany.
E-mail address: `gonzalez@math.uni-hannover.de`

Piotr Pokora, Instytut Matematyki, Pedagogical University of Cracow, Podchorążych 2, PL-30-084 Kraków, Poland.
Current Address: Institut für Algebraische Geometrie, Leibniz Universität Hannover, Welfengarten 1, D-30167 Hannover, Germany.
E-mail address: `piotrpkr@gmail.com, pokora@math.uni-hannover.de`